Math 434 Assignment 2

Due April 12

Assignments will be collected in class.

1. A set $X \subseteq \mathbb{N}$ is computable if there is a computer program in your favourite language which, on input n, outputs 0 is $n \in X$ and outputs 1 if $n \notin X$. (A computer program is a finite string of symbols.) Using an argument involving cardinality, prove that there is a set which is not computable.

2. For κ, λ infinite cardinals with $\lambda < \kappa$ prove that $|\{x \subseteq \kappa : |x| = \lambda\}| = \kappa^\lambda$.

3. A class function $f : ORD \to ORD$ is normal if it is order-preserving and continuous. Continuous means that for any limit ordinal λ we have that

$$f(\lambda) = \lim_{\alpha < \lambda} f(\alpha).$$

   (a) Given any normal class function f, prove that for all α, there is $\beta \geq \alpha$ such that $f(\beta) = \beta$.

   (b) Show that the function $\alpha \mapsto \omega \cdot \alpha$ is order-preserving and continuous.

   (c) 0 is the least fixed point of this function. Show that ω^ω is the next fixed point.

4. Prove that $2^{\aleph_0} \neq \aleph_\omega$.

5. Prove that if $\mathcal{U}$ is an ultrafilter on X, and $X = X_1 \cup \cdots \cup X_n$, then some X_i is in $\mathcal{U}$.

6. Prove that if an ultrafilter contains a finite set, then it is principal.

7. Let $\mathcal{U}$ be an ultrafilter. Let $(x_i)_{i \in \omega}$ be a sequence of real numbers in $[0,1]$. Show that there is a unique $x \in [0,1]$ such that for all $\epsilon > 0$, the set $\{i : |x - x_i| < \epsilon\}$ is in the ultrafilter. (We call x the $\mathcal{U}$-limit of this sequence; note that every sequence has a limit in this sense.)

8. In an election there is a finite set A of candidates and a countable (possibly finite) set V of voters. Let $L(A)$ be the set of linear orderings of A. A social welfare function is a function

$$F : L(A)^V \to L(A).$$

   That is, it takes as input an ordering, for each voter, on the candidates A (these are the votes), and outputs a linear order on A which is the outcome of the election.

   Three desirable conditions are:

Unanimity: If all of the voters enter the same ranking, then this is the outcome.

Independence of irrelevant alternatives: The ordering of two candidates a and b in the outcome only depends on the ordering of a and b by each voter, and not on how the voters ranked the other candidates.

Non-dictatorship There is no voter $v_0 \in V$ such that the outcome of the election is just the preferences of v_0.

We will prove in class that if F is a social choice function satisfying **Unanimity** and **Independence of irrelevant alternatives**, and there are at least three candidates, then there is an ultrafilter $\mathcal{U}$ on V such that $F((R_v)_{v\in V}) = S$ if and only if $\{v \in V : R_v = S\} \in \mathcal{U}$. In particular, if V is finite, then $\mathcal{U}$ is principal and so there there is a dictator.

If the set of voters is countably infinite, say $V = \omega$, use ultrafilters to show that there is a social choice function satisfying **Unanimity**, **Independence of Irrelevent Alternatives**, and with no **Dictator**.

So ultrafilters give a good voting scheme with infinite sets of voters.